

A curious differential calculus on the quantum disc and cones

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Abstract. A non-classical differential calculus on the quantum disc and cones is constructed and the associated integral is calculated.

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1. Introduction

The aim of this note is to present a two-dimensional differential calculus on the quantum disc algebra, which has no counterpart in the classical limit, but admits a well-defined (albeit different from the one in [2]) integral, and restricts properly to the quantum cone algebras. In this way the results of [3] are extended to other classes of non-commutative surfaces and to higher forms. The presented calculus is associated to an orthogonal pair of skew-derivations, which arise as a particular example of skew-derivations on generalized Weyl algebras constructed recently in [1]. It is also a fundamental ingredient in the construction of the Dirac operator on the quantum cone [6] that admits a twisted real structure in the sense of [5].

The reader unfamiliar with non-commutative differential geometry notions is referred to [4].

2. A differential calculus on the quantum disc

Let $0 < q < 1$. The coordinate algebra of the quantum disc, or the quantum disc algebra $\mathcal{O}(D_q)$ [8] is a complex $*$ -algebra generated by z subject to

$$z^*z - q^2zz^* = 1 - q^2. \quad (2.1)$$

To describe the algebraic contents of $\mathcal{O}(D_q)$ it is convenient to introduce a self-adjoint element $x = 1 - zz^*$, which q^2 -commutes with the generator of $\mathcal{O}(D_q)$, $xz = q^2zx$. A linear basis of $\mathcal{O}(D_q)$ is given by monomials $x^k z^l, x^k z^{*l}$. We view $\mathcal{O}(D_q)$ as a \mathbb{Z} -graded algebra, setting $\deg(z) = 1$,

$\deg(z^*) = -1$. Associated with this grading is the degree-counting automorphism $\sigma : \mathcal{O}(D_q) \rightarrow \mathcal{O}(D_q)$, defined on homogeneous $a \in \mathcal{O}(D_q)$ by $\sigma(a) = q^{2\deg(a)}a$. As explained in [1] there is an orthogonal pair of skew-derivations $\partial, \bar{\partial} : \mathcal{O}(D_q) \rightarrow \mathcal{O}(D_q)$ twisted by σ and given on the generators of $\mathcal{O}(D_q)$ by

$$\partial(z) = z^*, \quad \partial(z^*) = 0, \quad \bar{\partial}(z) = 0, \quad \bar{\partial}(z^*) = q^2 z, \quad (2.2)$$

and extended to the whole of $\mathcal{O}(D_q)$ by the (right) σ -twisted Leibniz rule. Therefore, there is also a corresponding first-order differential calculus $\Omega^1(D_q)$ on $\mathcal{O}(D_q)$, defined as follows.

As a left $\mathcal{O}(D_q)$ -module, $\Omega^1(D_q)$ is freely generated by one forms $\omega, \bar{\omega}$. The right $\mathcal{O}(D_q)$ -module structure and the differential $d : \mathcal{O}(D_q) \rightarrow \Omega^1(D_q)$ are defined by

$$\omega a = \sigma(a)\omega, \quad \bar{\omega} a = \sigma(a)\bar{\omega}, \quad d(a) = \partial(a)\omega + \bar{\partial}(a)\bar{\omega}. \quad (2.3)$$

In particular,

$$dz = z^*\omega = q^2\omega z^*, \quad dz^* = q^2 z\bar{\omega} = \bar{\omega} z, \quad (2.4)$$

and so, by the commutation rules (2.3),

$$\omega = \frac{q^{-2}}{1-q^2} (dz z - q^4 z dz), \quad \bar{\omega} = \frac{q^{-2}}{1-q^2} (z^* dz^* - q^2 dz^* z^*). \quad (2.5)$$

Hence $\Omega^1(D_q) = \{\sum_i a_i db_i \mid a_i, b_i \in \mathcal{O}(D_q)\}$, i.e. $(\Omega^1(D_q), d)$ is truly a first-order differential calculus not just a degree-one part of a differential graded algebra. The appearance of $q^2 - 1$ in the denominators in (2.5) indicates that this calculus has no classical (i.e. $q = 1$) counterpart.

The first-order calculus $(\Omega^1(D_q), d)$ is a $*$ -calculus in the sense that the $*$ -structure extends to the bimodule $\Omega^1(D_q)$ so that $(a\nu b)^* = b^* \nu^* a^*$ and $(da)^* = d(a^*)$, for all $a, b \in \mathcal{O}(D_q)$ and $\nu \in \Omega^1(D_q)$, provided $\omega^* = \bar{\omega}$ (this choice of the $*$ -structure justifies the appearance of q^2 in the definition of $\bar{\partial}$ in equation (2.2)). From now on we view $(\Omega^1(D_q), d)$ as a $*$ -calculus, which allows us to reduce by half the number of necessary checks.

Next we aim to show that the module of 2-forms $\Omega^2(D_q)$ obtained by the universal extension of $\Omega^1(D_q)$ is generated by the anti-self-adjoint 2-form¹

$$\mathbf{v} = \frac{q^{-6}}{q^2 - 1} (\omega^* \omega + q^8 \omega \omega^*), \quad \mathbf{v}^* = -\mathbf{v} \quad (2.6)$$

and to describe the structure of $\Omega^2(D_q)$. By (2.3), for all $a \in \mathcal{O}(D_q)$,

$$\mathbf{v} a = \sigma^2(a) \mathbf{v}. \quad (2.7)$$

Combining commutation rules (2.3) with the relations (2.4) we obtain

$$z^* dz = q^2 dz z^*, \quad dz z - q^4 z dz = q^2 (1 - q^2) \omega, \quad (2.8)$$

¹One should remember that the $*$ -conjugation takes into account the parity of the forms; see [9].

and their $*$ -conjugates. The differentiation of the first of equations (2.8) together with (2.3) and (2.1) yield

$$\omega\omega^* = (1-x)\mathbf{v}, \quad \omega^*\omega = q^6(q^2x-1)\mathbf{v}, \quad (2.9)$$

which means that $\omega\omega^*$ and $\omega^*\omega$ are in the module generated by \mathbf{v} . Next, by differentiating $\omega z^* = q^{-2}z^*\omega$ and $\omega z = q^2z\omega$ and using (2.4) and (2.3) one obtains

$$d\omega z^* = q^{-2}z^*d\omega + z(\omega^*\omega + q^4\omega\omega^*), \quad d\omega z = q^2z d\omega + (q^2 + q^{-2})z^*\omega^2. \quad (2.10)$$

The differentiation of $dz = z^*\omega$ yields

$$z^*d\omega = -q^2z\omega^*\omega. \quad (2.11)$$

Multiplying this relation by z from left and right, and using commutation rules (2.1) and (2.3) one finds that $(1-x)d\omega = q^{-4}z^*d\omega z$. Developing the right hand side of this equality with the help of the second of equations (2.10) we find

$$d\omega = \frac{1+q^{-4}}{q^2-1}z^{*2}\omega^2. \quad (2.12)$$

Combining (2.10) with (2.12) we can derive

$$z^{*3}\omega^2 = -z\frac{q^8}{q^4+1}(\omega^*\omega + q^4\omega\omega^*). \quad (2.13)$$

The multiplication of (2.13) by z^3 from the left and right and the usage of (2.1), (2.3) give

$$(1-x)(1-q^{-2}x)(1-q^{-4}x)\omega^2 = -\frac{q^8}{q^4+1}z^4(\omega^*\omega + q^4\omega\omega^*), \quad (2.14a)$$

$$(1-q^2x)(1-q^4x)(1-q^6x)\omega^2 = -\frac{q^8}{q^4+1}z^4(\omega^*\omega + q^4\omega\omega^*). \quad (2.14b)$$

Comparing the left hand sides of equations (2.14), we conclude that

$$x\omega^2 = 0 = \omega^2x \quad \text{and, by } * \text{-conjugation, } x\omega^{*2} = 0 = \omega^{*2}x, \quad (2.15)$$

and hence in view of either of (2.14)

$$\omega^2 = -\frac{q^8}{q^4+1}z^4(\omega^*\omega + q^4\omega\omega^*). \quad (2.16)$$

By (2.9), the right hand side of (2.16) is in the module generated by \mathbf{v} , and so is ω^2 and its adjoint ω^{*2} . Thus, the module $\Omega^2(D_q)$ spanned by all products of pairs of one-forms is indeed generated by \mathbf{v} .

Multiplying (2.12) and (2.11) by x and using relations (2.15) we obtain

$$xz\omega^*\omega = 0 = \omega^*\omega xz. \quad (2.17)$$

Following the same steps but now starting with the differentiation of $dz^* = q^2z\omega^*$ (see (2.4)), we obtain the complementary relation

$$xz\omega\omega^* = 0 = \omega\omega^*xz. \quad (2.18)$$

In view of the definition of \mathbf{v} , (2.17) and (2.18) yield $xzv = 0 = vxz$. Next, the multiplication of, say, the first of these equations from the left and right

by z^* and the use of (2.1) yield $x(1-x)v = 0$ and $x(1-q^2x)v = 0$. The subtraction of one of these equations from the suitable scalar multiple of the other produces the necessary relation

$$xv = 0 = vx, \quad (2.19)$$

which fully characterises the structure of $\Omega^2(D_q)$ as an $\mathcal{O}(D_q)$ -module generated by v . In the light of (2.19), the \mathbb{C} -basis of $\Omega^2(D_q)$ consists of elements vz^n , vz^{*m} , and hence, for all $w \in \Omega^2(D_q)$, $wx = xw = 0$, i.e., $\Omega^2(D_q)$ is a torsion (as a left and right $\mathcal{O}(D_q)$ -module). Since $\mathcal{O}(D_q)$ is a domain and $\Omega^2(D_q)$ is a torsion, the dual of $\Omega^2(D_q)$ is the zero module, hence, in particular $\Omega^2(D_q)$ is not projective. Again by (2.19), the annihilator of $\Omega^2(D_q)$,

$$\text{Ann}(\Omega^2(D_q)) := \{a \in \mathcal{O}(D_q) \mid \forall w \in \Omega^2(D_q), aw = wa = 0\},$$

is the ideal of $\mathcal{O}(D_q)$ generated by x . The quotient $\mathcal{O}(D_q)/\text{Ann}(\Omega^2(D_q))$ is the Laurent polynomial ring in one variable, i.e. the algebra $\mathcal{O}(S^1)$ of coordinate functions on the circle. When viewed as a module over $\mathcal{O}(S^1)$, $\Omega^2(D_q)$ is free of rank one, generated by v . Thus, although the module of 2-forms over $\mathcal{O}(D_q)$ is neither free nor projective, it can be identified with sections of a trivial line bundle once pulled back to the (classical) boundary of the quantum disc.

With (2.19) at hand, equations (2.9), (2.16), (2.12) and their $*$ -conjugates give the following relations in $\Omega^2(D_q)$

$$d\omega = q^8 z^2 v, \quad d\omega^* = -z^{*2} v, \quad \omega\omega^* = v, \quad \omega^*\omega = -q^6 v, \quad (2.20a)$$

$$\omega^2 = q^{12} \frac{q^2 - 1}{q^4 + 1} z^4 v, \quad \omega^{*2} = q^{-4} \frac{q^2 - 1}{q^4 + 1} z^{*4} v. \quad (2.20b)$$

One can easily check that (2.20), (2.19) and (2.7) are consistent with (2.3) with no further restrictions on v . Setting $\Omega^n(D_q) = 0$, for all $n > 2$, we thus obtain a 2-dimensional calculus on the quantum disc.

3. Differential calculus on the quantum cone

The quantum cone algebra $\mathcal{O}(C_q^N)$ is a subalgebra of $\mathcal{O}(D_q)$ consisting of all elements of the \mathbb{Z} -degree congruent to 0 modulo a positive natural number N . Obviously $\mathcal{O}(C_q^1) = \mathcal{O}(D_q)$, the case we dealt with in the preceding section, so we may assume $N > 1$. $\mathcal{O}(C_q^N)$ is a $*$ -algebra generated by the self-adjoint $x = 1 - zz^*$ and by $y = z^N$, which satisfy the following commutation rules

$$xy = q^{2N} yx, \quad yy^* = \prod_{l=0}^{N-1} (1 - q^{-2l} x), \quad y^*y = \prod_{l=1}^N (1 - q^{2l} x). \quad (3.1)$$

The calculus $\Omega(C_q^N)$ on $\mathcal{O}(C_q^N)$ is obtained by restricting of the calculus $\Omega(D_q)$, i.e. $\Omega^n(C_q^N) = \{\sum_i a_0^i d(a_1^i) \cdots d(a_n^i) a_{n+1}^i \mid a_k^i \in \mathcal{O}(C_q^N)\}$. Since d is a degree-zero map $\Omega(C_q^N)$ contains only these forms in $\Omega(D_q)$, whose \mathbb{Z} -degree is a multiple of N . We will show that all such forms are in $\Omega(C_q^N)$. Since $\deg(\omega) = 2$, $\deg(\omega^*) = -2$ and $\deg(v) = 0$, this is equivalent to

$$\Omega^1(C_q^N) = \mathcal{O}(D_q)_{\overline{2}} \omega \oplus \mathcal{O}(D_q)_{\overline{2}} \omega^*, \quad \Omega^2(C_q^N) = \mathcal{O}(C_q^N) v,$$

where $\mathcal{O}(D_q)_{\overline{s}} = \{a \in \mathcal{O}(D_q) \mid \deg(a) \equiv s \pmod{N}\}$.

As an $\mathcal{O}(C_q^N)$ -module, $\mathcal{O}(D_q)_{\overline{-2}}$ is generated by z^{N-2} and z^{*2} , hence to show that $\mathcal{O}(D_q)_{\overline{-2}}\omega \subseteq \Omega^1(C_q^N)$ suffices it to prove that $z^{N-2}\omega, z^{*2}\omega \in \Omega^1(C_q^N)$. Using the Leibniz rule one easily finds that

$$dy = ([N; q^2] - q^{-2N+4} [N; q^4] x) z^{N-2}\omega,$$

where $[n; s] := \frac{s^n - 1}{s - 1}$. Hence, in view of (2.1) and (2.3),

$$y^* dy = [N; q^2] \left(1 - q^4 \frac{[N; q^4]}{[N; q^2]} x \right) \prod_{l=3}^N (1 - q^{2l} x) z^{*2}\omega, \quad (3.2a)$$

$$dy y^* = q^{-2N} [N; q^2] \left(1 - q^{-2N+4} \frac{[N; q^4]}{[N; q^2]} x \right) \prod_{l=0}^{N-3} (1 - q^{-2l} x) z^{*2}\omega. \quad (3.2b)$$

The polynomial in x on the right hand side of (3.2a) has roots in common with the polynomial on the right hand side of (3.2b) if and only if there exists an integer $k \in [-2N + 2, -N - 1] \cup [2, N - 1]$ such that

$$q^{2k}(q^{2N} + 1) = q^2 + 1. \quad (3.3)$$

Equation (3.3) is equivalent to $q^2 [N + k - 1; q^2] + [k; q^2] = 0$, with the left hand side strictly positive if $k > 0$ and strictly negative if $k \leq -N$. So, there are no solutions within the required range of values of k . Hence the polynomials (3.2a), (3.2b) are coprime, and so there exists a polynomial (in x) combination of the left hand sides of equations (3.2) that gives $z^{*2}\omega$. This combination is an element of $\Omega^1(C_q^N)$ and so is $z^{*2}\omega$. Next,

$$z^{*2}\omega y = q^{2N}(1 - q^2 x)(1 - q^4 x) z^{N-2}\omega,$$

$$y z^{*2}\omega = (1 - q^{-2N+4} x)(1 - q^{-2N+2} x) z^{N-2}\omega,$$

so again there is an x -polynomial combination of the left hand sides (which are already in $\Omega^1(C_q^N)$) giving $z^{N-2}\omega$. Therefore, $\mathcal{O}(D_q)_{\overline{-2}}\omega \subseteq \Omega^1(C_q^N)$. The case of $\mathcal{O}(D_q)_{\overline{2}}$ follows by the $*$ -conjugation.

Since $z^2\omega^*, z^{*2}\omega$ are elements of $\Omega^1(C_q^N)$,

$$\Omega^2(C_q^N) \ni z^2\omega^* z^{*2}\omega = q^{-4}(1 - x)(1 - q^{-2}x)\omega^*\omega = -q^2 v, \quad (3.4)$$

by the quantum disc relations and (2.20) and (2.19). Consequently, $v \in \Omega^2(C_q^N)$. Therefore, $\Omega(C_q^N)$ can be identified with the subspace of $\Omega(D_q)$, of all the elements whose \mathbb{Z} -degree is a multiple of N .

4. The integral

Here we construct an algebraic integral associated to the calculus constructed in Section 2. We start by observing that since σ preserves the \mathbb{Z} -degrees of elements of $\mathcal{O}(D_q)$ and ∂ and $\bar{\partial}$ satisfy the σ -twisted Leibniz rules, the definition (2.2) implies that ∂ lowers while $\bar{\partial}$ raises degrees by 2. Hence, one can equip $\Omega^1(D_q)$ with the \mathbb{Z} -grading so that d is the degree zero map,

provided $\deg(\omega) = 2$, $\deg(\omega^*) = -2$. Furthermore, in view of the definition of σ , one easily finds that

$$\sigma^{-1} \circ \partial \circ \sigma = q^4 \partial, \quad \sigma^{-1} \circ \bar{\partial} \circ \sigma = q^{-4} \bar{\partial}, \quad (4.1)$$

i.e. ∂ is a q^4 -derivation and $\bar{\partial}$ is a q^{-4} -derivation. Therefore, by [7], $\Omega(D_q)$ admits a divergence, for all right $\mathcal{O}(D_q)$ -linear maps $f : \Omega^1(D_q) \rightarrow \mathcal{O}(D_q)$, given by

$$\nabla_0(f) = q^4 \partial(f(\omega)) + q^{-4} \bar{\partial}(f(\omega^*)). \quad (4.2)$$

Since the $\mathcal{O}(D_q)$ -module $\Omega^2(D_q)$ has a trivial dual, ∇_0 is flat. Recall that by the *integral* associated to ∇_0 we understand the cokernel map of ∇_0 .

Theorem 4.1. *The integral associated to the divergence (4.2) is a map $\Lambda : \mathcal{O}(D_q) \rightarrow \mathbb{C}$, given by*

$$\Lambda(x^k z^l) = \lambda \frac{[k+1; q^2]}{[k+1; q^4]} \delta_{l,0}, \quad \text{for all } k \in \mathbb{N}, l \in \mathbb{Z}, \quad (4.3)$$

where, for $l < 0$, z^l means z^{*-l} and $\lambda \in \mathbb{C}$.

Proof. First we need to calculate the image of ∇_0 . Using the twisted Leibniz rule and the quantum disc algebra commutation rules (2.1), one obtains

$$\partial(x^k) = -q^{-2} [k; q^4] x^{k-1} z^{*2}. \quad (4.4)$$

Since $\partial(z^*) = 0$, (4.4) means that all monomials $x^k z^{*l+2}$ are in the image of ∂ hence in the image of ∇_0 . Using the $*$ -conjugation we conclude the $x^k z^{l+2}$ are in the image of $\bar{\partial}$ hence in the image of ∇_0 . So Λ vanishes on (linear combinations of) all such polynomials. Next note that

$$\partial(z^2) = (q^2 + 1) - (q^4 + 1)x, \quad (4.5)$$

hence

$$\partial(z^* z^2 - q^4 z^2 z^*) = (1 - q^4) z^*, \quad \partial(z^* z^2 - q^2 z^2 z^*) = (1 - q^2)(1 + q^4) x z^*.$$

This means that z^* and xz^* are in the image of ∂ , hence of ∇_0 . In fact, all the $x^k z^*$ are in this image which can be shown inductively. Assume $x^k z^* \in \text{Im}(\partial)$, for all $k \leq n$. Then using the twisted Leibniz rule, (4.4) and (4.5) one finds

$$\partial(x^n z^2) = -q^2 [N; q^4] x^{n-1} + (q^2 + 1) [n+1; q^4] x^n - [n+2; q^4] x^{n+1}. \quad (4.6)$$

Since $\partial(z^*) = 0$, equation (4.6) implies that $\partial(x^n z^2 z^*)$ is a linear combination of monomials $x^{n-1} z^*$, $x^n z^*$ and $x^{n+1} z^*$. Since the first two are in the image of ∂ by the inductive assumption, so is the third one. Therefore, all linear combinations of $x^k z^*$ and $x^k z$ (by the $*$ -conjugation) are in the image of ∇_0 .

Put together all this means that Λ vanishes on all the polynomials $\sum_{k,l=1}^n (c_{kl} x^k z^l + c'_{kl} x^k z^{*l})$. The rest of the formula (4.3) can be proven by induction. Set $\lambda = \Lambda(1)$. Since Λ vanishes on all elements in the image of ∇_0 , hence also in the image of ∂ , the application of Λ to the right hand side of (4.4) confirms (4.3) for $k = 1$. Now assume that (4.3) is true for all $k \leq n$.

Then the application Λ to the right hand side of (4.6) followed by the use of the inductive assumption yields

$$\begin{aligned} [n+2; q^4] \Lambda(x^{n+1}) &= q^2 [N; q^4] \Lambda(x^{n-1}) - (q^2 + 1) [n+1; q^4] \Lambda(x^n) \\ &= \lambda((q^2 + 1) [n+1; q^2] - q^2 [n; q^2]) = \lambda [n+2; q^2]. \end{aligned}$$

Therefore, the formula (4.3) is true also for $n+1$, as required. \square

The restriction of Λ to the elements of $\mathcal{O}(D_q)$, whose \mathbb{Z} -degree is a multiple of N gives an integral on the quantum cone $\mathcal{O}(C_q^N)$.

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